The Theory of School Arithmetic: Fractions[1]

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Introduction

There are at least two different perspectives on arithmetic in elementary school, each embodied by a corpus of content knowledge.

In one perspective, the main purpose of arithmetic is that students learn to compute the four basic operations with whole numbers and fractions, e.g., to give the answer 3 to the question $1 + 2 = ?$, or the answer $\frac{1}{6}$ to $\frac{1}{2} \times \frac{1}{3} = ?$. The content knowledge that embodies this perspective includes only computational procedures. Apparently, what students learn by studying this body of content knowledge is very similar to what a calculator does. When calculators became common, the necessity of having young students master this knowledge became doubtful.

A second perspective is that students use arithmetic to acquire a sound foundation for their future—for everyday life when they grow up, as well as further study of mathematics. The corresponding content knowledge has two main components. One is proficiency in computing. Gained by following conventional procedures as well as using shortcuts,[2] it is considered not only as a skill needed in everyday life, but also the "substance" to foster and sharpen students' understanding of numerical concepts such as those described in this article. Also, proficiency in computing greatly eases mental load later when students approach sophisticated mathematical problems.

The other component is the ability to solve word problems by using an "arithmetic equation" or a "quasi-equation."[3] Students learn how to formulate an arithmetic equation to display their approaches to solving a word problem, from one-step whole number problems through multi-step problems with fractions. This ability, consisting of skills and the habit of analyzing a quantitative relationship and representing it with a quasi-equation, prepares students to move on to their later learning of algebra, geometry, etc. Also, the process of developing this ability serves as the "substance" to foster students' logical and abstractive

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[1] This article is a continuation of "The Theory of School Arithmetic: Whole Numbers" (Ma & Kessel, 2018) but can be read as an independent article.

[2] Following the conventional procedure is not always the "best" or the "fastest" way to compute. For example, to add 27 to 98, following the conventional procedure we first add 7 to 8 and get 15, carry one ten in 15 to tens place. Then at the tens place we add 2 to 9, and carrying the 1, get 12 tens. We carry the one hundred in 12 tens to the hundreds place, and finally get the sum as 125. Or, bypassing this conventional way, we just add 2 from 27 to 98 and get 100. Then add the remaining 25 to 100 and get 125. This non-conventional way of computing is not an odd, wild, and inexplicable approach. Rather, it is justified by the theory of school arithmetic.

[3] The definition of "equation" in different dictionaries varies. There are two characteristics of an equation that all may agree on: 1) it is horizontally presented; 2) it has three parts: two equivalent mathematical expressions connected by the equal sign, for example, $x + 5 = 9$. However, sometimes only two of these parts are used, for example, $54 + 21 =$. We call an expression followed by an equal sign a "quasi-equation." It is the "stepping stone" for children's use of equations in their future learning.

thinking. Word problems in school arithmetic are usually presented with situations related to everyday life. An important task of this type of presentation is to trigger students' intellectual curiosity and to lead their minds beyond everyday life.[4]

These two components are woven together by what we call "the theory of school arithmetic." An important stage in this theory's development occurred during the rise of universal education in the United States in the second half of the nineteenth century. During this rapidly spreading movement, mathematical scholars wrote textbooks for elementary school students. Within these textbooks, "arithmetic was increasingly formalized into a logical system" (Bidwell & Clason, 1970, p. 1). This effort endowed "merchant arithmetic"—a collection of rules for computing commercial transactions—with the potential to foster logical thinking and reformed it as a school subject. However, within the United States this logical system was ignored during the Progressive era and abandoned during the New Math.[5]

Outside the United States, it survived, and was improved and enriched in a few other countries, for example, East Asian countries where students perform well on international surveys such as TIMSS and PISA. Indeed, we believe that this logical system for school arithmetic is the root of what Ma called "Profound Understanding of Fundamental Mathematics."

In a previous article, "The theory of school arithmetic: Whole numbers" (Ma & Kessel, 2018), we described the part of this logical system that concerns whole numbers and base-ten notation for whole numbers. In the present article, we discuss how this logical system is extended to include fractions.

In our previous article we stated that the theory of school arithmetic has two important features:

- a system of definitions and axioms modeled on the definitions, postulates, and common notions of Euclid's *Elements*. These make relationships among operations explicit and afford explanations for algorithms and computations in terms of a small number of assumptions. This article focuses on the definitions rather than the axioms. (These axioms are compensation, and the commutative, associative, and distributive properties.)

- horizontal equations and expressions. Relationships among operations may be expressed symbolically by a series of equations, e.g., if $1 + 2 = 3$, then $3 - 1 = 2$. Unlike vertical columns, horizontal expressions allow students to solve sophisticated problems, showing a solution as a single arithmetical expression given in terms of the operations used to obtain it. These are illustrated by examples at the end of this article.

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[4] There have been two main motivations for people to pursue mathematical knowledge, one is to solve problems in everyday life, the other is to satisfy intellectual curiosity. Both are significant in terms of the development of the discipline. In education, we should make sure that neither is ignored.

[5] For discussion of the abandonment of the system during the New Math, see Ma (2013).

The theory also has rules that are consequences of one or more definitions. For example, an instance of the rule of like numbers is that 2 apples and 3 oranges cannot be added.

Logical features of the theory. As in the *Elements,* definitions are given in an order that is logical in the sense that if one definition depends on another, then it appears after that definition. For example, because the definition of addition depends on the definition of sum, the definition of addition is given after the definition of sum. Thus, these definitions are not are not circular as with dictionary definitions, but follow Pascal's 1655 dictum "In definitions use only terms which are perfectly clear in themselves or which have been previously defined" (as quoted in Young, 1920, p. 190).

A longstanding tradition in philosophy advocates simplicity in theoretical formulations of mathematics and science (Baker, 2016). Some examples are Ockham's Razor (sometimes known as the Principle of Parsimony) and a statement attributed to Einstein "Everything should be made as simple as possible, but no simpler" (for further examples in a variety of fields and eras, see Baker, 2016). For mathematical theories, simplicity is often referred to as "parsimony" and often focuses on use of the fewest possible assumptions and on use of definitions that are general enough to include all cases under consideration. The theory presented in this article is parsimonious in at least two ways.

- Rather than being given independently, the definitions of operations are given in terms of two basic quantitative relationships: sum and addends; product, multiplier, and multiplicand. For example, the definition of addition is given in terms of the relationship of sum and addends.

- The two basic quantitative relationships for whole numbers and fractions are the same.

Psychological features of the theory. In their collection of readings about mathematics education in the United States, Bidwell and Clason remark, "In the mid-nineteenth century school mathematics was still the science of quantity and geometry the science of space. Basing mathematics on quantity allowed mathematical and psychological theory to mesh; one initially learned mathematics from quantitative reality" (1970, p. 2). They note that in practice, this meant using manipulative objects such as cubes, beans, corn, and buttons in early number work.

In the theory, this emphasis on quantity is reflected in the definition of a number as a collection of units[6] and the notion of concrete numbers (e.g., 3 beans, 4 buttons, or 5 inches) as opposed to abstract numbers (e.g., 3, 4, 5). The distinction between concrete and abstract numbers is similar to the modern-day distinction between quantities with and without dimensions (see *The International System of Units (SI)*, pp. 13–14). In the classroom, this distinction is sometimes described in terms of units: "5 inches" has units; "5" does not.

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[6] This definition allows 1 to be a number, thus is a modification of the definition of number in Euclid's *Elements*.

A feature of the theory that is different from a modern-day US viewpoint is the notion that abstract numbers mirror concrete numbers in having "number units" that act like units of measurement. A positive integer is a collection of ones and can be "measured" in terms of ones, tens, hundreds, etc. A positive rational number can be "measured" in terms of "fractional units" or "decimal fraction units." Today, we can see the system of units for abstract numbers as mirroring the system of metric units for length (or for mass). For example, in the metric system there is a basic unit (meter) and "many as one" units (e.g., 10 meters make 1 decameter) and "fractional units" (e.g., 1/10 meter is 1 decimeter). The corresponding number units are ones, tens, and tenths.

In the theory, we have used "unit value" in stating rules for number units. Rules such as "Only like numbers (i.e., numbers with the same units) may be added" are introduced for concrete numbers, followed by their analogues for computation with abstract numbers, e.g., "Only digits with like unit value may be added." Rules are introduced first for whole numbers (concrete and abstract), then for fractions.

Organization of this article. The theory has two sections: one on whole numbers and one on fractions. This article focuses on the second section. We begin by summarizing the section on whole numbers, giving the definitions and rules from our article on whole numbers (for a more detailed discussion, see Ma & Kessel, 2018). The remainder of this article discusses the section of the theory that pertains to fractions. Although definitions and rules for whole remain the same for fractions, associated concepts expand. These and other pedagogical features are mentioned in pedagogical remarks that follow definitions and rules.

Note that the theory described in this article *underlies* the type of school arithmetic in which students explore and represent relationships among the four operations. Although some nineteenth-century school textbooks presented definitions, rules, and axioms like those in this article, more recent elementary textbooks do not present them as formulated here. Moreover, some distinctions need not be made for students. For example, students need not make explicit distinctions between numbers and numerals or use terms such as "digits with like unit value." Instead, they speak of "the digits of a number" rather than "the digits of a numeral" or "digits with the same units." Such distinctions are needed in the theory to express connections between numbers and operations and the notation used to represent and compute them.

I. General definitions: Whole numbers

Units, numbers, abstract, concrete, and like numbers. A single thing, or a one, is called a *unit* or *unit one*. The name of the thing is the name of the unit. "One" is considered an unnamed unit. A group of things or a group of units, if considered as a single thing or one, is also called a *unit*. In this definition, we see two types of unit. The first type we call "one-as-one unit" and the second "many-as-one unit."

A *number* is a unit or a collection of units.

An *abstract number* is a number whose units are not named.

A *concrete number* is a collection of units with the same name.

If two concrete numbers each have units with the same name, they are called *like numbers*.

Although the concept is called "unit," use of the terms "unit one" and "one" in teaching helps to connect "unit" with students' conception of "one."

There is a subtle conceptual difference between viewing something as *being* a collection of units and being *described* in terms of units. This difference in views is analogous to two views of measurement. For example, when a book is described as "10 inches long," one might envision a collection of 10 identical objects, each called "an inch," laid end-to-end along the side of a book. Or, more abstractly, the 10 identical inches are represented on a ruler as identical lengths, but not identical objects, instead the first length is labeled 1, the second 2, and so on.

II.   Fractions
(Introducing a new set of numbers, expanding the concept of number)

Fractional unit. If one unit is partitioned in several equal parts, each of these equal parts is called a *fractional unit*. Any two of these equal parts are *like fractional units.* Thus, 1/3 and 1/3 are like fractional units as are 1/3 apple and 1/3 apple.
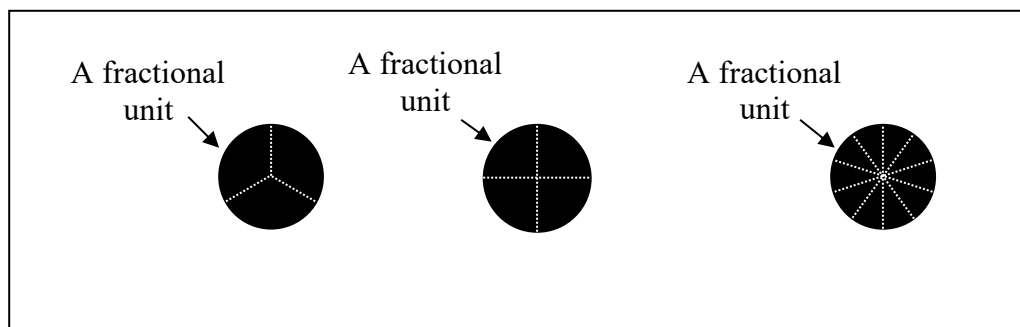

Fig. 5.1 Fractional unit

A fractional unit is an abstract concept. In children's everyday life, except for "half" or "half of half" there are few opportunities for them to experience a concrete number that is a fractional unit. From fractional units, a new set of numbers is generated: fractions.

A fractional unit, if considered as a single thing or one, is also called a unit or a unit one. A group of fractional units, if considered as a single thing or one, is also called a unit or a unit one.

The theory of arithmetic begins with unit one. The concept of unit one increases in abstraction several times (see Table 1). The concept of many-as-one unit is one increase in abstraction. The concept of a single fractional unit is another increase in abstraction.

Considering several fractional units as one unit is yet another increase in abstraction. Each of these types of unit has its own role in the theory.

Table 1. The main types of units

Type of unit	Where unit is first used
One as one	Addition and subtraction of whole numbers
Many integral units as one	Multiplication and division of whole numbers
Fractional unit as one	Concept of fraction, addition and subtraction of fractions
Many fractional units as one	Multiplication and division of fractions

Table 2. Unit values represented in notation systems

Type of unit value	Where used
Nonnegative integer powers of ten	Base-ten notation for whole numbers
Negative integer powers of ten	Base-ten notation for fractions
Parts of one	Numerator–denominator notation for fractions

Whole numbers and fractional numbers. One unit or a collection of like integral units is called a *whole number*. A *fractional number* (or *fraction*) is a fractional unit or a collection of like fractional units.[7] (For example, one seventh, three sevenths, and eight sevenths are all fractions.)

It is only after the concept of fraction is introduced, that there is a need for the terms whole number and integral unit.

In both cases, whole number and fractional number, a number is a collection of like units. Because fractional units are a new type of unit which (unlike many-as-one units) cannot be converted to integral units, this creates a new set of numbers.

In this way, the concept of number is expanded.

In human history, the natural numbers were the first type of numbers used. The first expansion was to fractions. Introducing a new type of numbers indicates use of the earlier numbers in a skillful way but also indicates awareness of their limitations. Use of a new type of number is significant in the discipline of mathematics.

School arithmetic reflects this historical trajectory. For elementary students, after they master operations and the two basic quantitative relationships with whole numbers, the experience of learning operations with fractions affords the opportunity to understand why introducing a new type of number is interesting and necessary.

Unit of a fraction and its value. A fraction is a collection of like fractional units. Any one of those units is the *unit* of the fraction. (For example, the unit of 3/5 is 1/5.)

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[7] "By employing the term '*fractional units*,' the same principles are made applicable to fractional numbers, for, all fractions are but collections of fractional units, these units having a known relation to 1" (Davies, 1857, p. iii).

The *unit value* of a fraction is equal to its fractional unit. (For example, the unit value of 3/5 is 1/5 and the unit value of 4/7 is 1/7.)

There is another term in mathematics, unit fraction. Although unit fraction and fractional unit have similar names and are written symbolically in the same way, the associated concepts are different. The concept of fractional unit emphasizes the concept of unit. It is a special type of unit from which fractional numbers are created. The term unit fraction emphasizes the concept of fraction. It is a type of fractional number. The concepts of fractional unit and unit fraction are not the same. In school arithmetic, the concept of fractional unit is very important, while the term unit fraction can be ignored.[8]

Notation, numerator, and denominator. Any fractional unit is the result of the partition of a unit (1, 1 apple, 1 inch, etc.) into equal parts. The fractional unit is written $1/n$ (or $1/n$ apple, $1/n$ inch, etc.), where n is the number of parts in the partition. The number n is called the *denominator* of the fraction.

The number of fractional units is indicated by the *numerator* of the fraction. One seventh is written 1/7, three sevenths is written 3/7.

Within the whole numbers, the result of division is not always defined, for example, there is no whole number that could be the quotient of a given whole number divided by a larger whole number, e.g., 3 ÷ 5. Because of this, remainders occur in division of whole numbers when "number" means only "whole number." As shown in this section, expanding the meaning of number to include fractions makes the result of division defined for any two whole numbers or any two fractions. (Note that zero is not one of the whole numbers under consideration.)

Fractions with the same denominators. Fractions with the same denominators have fractional units of the same value.

Fractions with the same denominator are very important when computing sums and differences of fractions.

When learning fractions, students will work with different types of fractions such as proper fractions, improper fractions, mixed numbers, and complex fractions. These different forms are more important for computation. They are not necessary concepts of the theory, so will not be discussed in detail here.

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[8] A unit fraction is a fraction with a 1 in its numerator. In ancient Egypt, every fraction was represented as a sum of unit fractions. For example:

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20} = \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10}$$

In modern mathematics, a unit fraction is also seen as the reciprocal of a whole number and plays an important role.

III.   Addition and subtraction of natural numbers[9]

The sum of two numbers. The *sum* of two numbers is a third number which contains as many units as the other two numbers taken together.

Addition. The operation of finding the sum of two numbers is called *addition*.

Addends. The two numbers summed are called *addends*.

The rule of like numbers for addition. When two addends are concrete numbers, they must be like numbers. Their sum and the two addends are like numbers.

> This rule is a consequence of the definitions of number, sum, and addition. Because a concrete number is defined as a collection of units with the same name, collecting the units of two unlike concrete numbers or of a concrete number and an abstract number does not result in a number.

> Another consequence of this rule and conventions for notation is that a numeral can be written as the sum of numerals each of which represents the value of exactly one type of base-ten unit, e.g., $21 = 20 + 1$.

The rule of like unit value for addition. When computing a sum, only digits of like unit value can be added.

> This rule is the analogue for numerals of the rule of like numbers.

Subtraction. If a sum and one addend are known, the operation of finding the unknown addend is called *subtraction*.

Minuend, subtrahend, difference. The known sum in subtraction is called the *minuend*. The known addend is called the *subtrahend*. The unknown addend, which is the result of the operation of subtraction, is called the *difference*.

The rule of like numbers and the rule of like units for addition applied to subtraction. When minuend and subtrahend are concrete numbers, they must be like numbers. (This is a consequence of the definitions of number and product.) Their difference, minuend, and subtrahend are also like numbers. When computing a difference, only digits of like unit value can be subtracted.

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[9] Natural numbers do not include zero.

<center>**IV.   Addition and subtraction of fractions**
(the definitions and axioms remain the same)</center>

The definitions of addition and subtraction of fractions are the same as those for whole numbers.

These definitions are made in terms of the basic quantitative relationship "sum of two numbers": The *sum* of two numbers is a third number which contains as many units as the other two numbers taken together. Here "number" includes whole numbers and fractions, and "unit" includes integral and fractional units. Note that defining a fraction to be a collection of fractional units rather than a collection of unit fractions allows the definitions of addition and subtraction to be the same for whole numbers and for fractions.

The rule of like numbers for addition and subtraction of concrete fractions is the same as that for whole numbers.

The rule of like unit value for addition and subtraction of abstract fractions. When adding and subtracting fractions, only fractions with the same unit value can be added and subtracted. In other words, when adding and subtracting fractions, only fractions with the same denominators can be added and subtracted.

Fractions with unlike denominators must be converted to like denominators in order to be added or subtracted.

When the concept of number is expanded to include fractions, the definition of addition and subtraction is the same as that for whole numbers. The axioms (i.e., compensation, and the commutative, associative, and distributive properties) and rules of computation (e.g., the rules of like numbers and like units) remain the same for addition and subtraction.

Teaching students to calculate sums and differences of fractions is one of the hardest tasks in teaching. Its difficulty can be decreased if the rule of like numbers for addition and subtraction is emphasized when students learn operations with whole numbers.

Converting fractions with unlike denominators to those with like denominators involves complicated computations with its own terminology. These terms (prime number, composite number, greatest common divisor, least common multiple) are not directly related to the theory, and will not be discussed here.

The numbers in whole-number word problems are mostly concrete numbers with units that refer to physical objects (e.g., 3 apples), but those in word problems for addition and subtraction of fractions refer mostly to less tangible objects. For example:

   a.   A group of students are doing a project together. They finished 1/7 of the project on the first day, 2/7 of the project on the second day. How much did they do on these two days? How much of the project is left to be done?

     b.   Bill and Cathy each borrow a copy of the same book. Bill read 2/5 of the book. Cathy read 3/8 of the book. Who read more? How much more?

After fractions are introduced, the next stage of learning involves working mainly with quantitative relationships involving numbers with units that may be more abstract in two ways: the units refer to less tangible objects (e.g., portions of a project finished or a book read) and may be fractional units (e.g., 1/7) or many-as-one fractional units (2/7).

V.   Multiplication and division of natural numbers

The product of two numbers. The *product* of two numbers is a third number which contains as many units as one number taken as many times as the units in the other.

Multiplication. The operation of finding the product of two numbers is called *multiplication*.

Multiplicand, multiplier, and factors. *Multiplicand* is the number to be taken. It is the first term in a multiplication expression. It is the term to the left of the multiplication sign. The *multiplier* is how many times the multiplicand is taken. It is the term to the right of the multiplication sign and is always an abstract number. When both multiplicand and multiplier are abstract numbers, they are also called *factors*.

The distinction between multiplicand and multiplier does not remain throughout elementary mathematics. Initially, it is important because it helps students to be aware of the new many-as-one unit (each copy of the multiplicand), helping them to expand their conception of unit.

In a tape diagram, the many-as-one unit (i.e., the multiplicand) is depicted as a length. The multiplier indicates the number of those lengths in the diagram. Illustrating multiplication with a tape diagram rather than an array of dots distinguishes multiplier and multiplicand, and may help students to "unitize" the multiplicand, that is, see the multiplicand as a single unit rather than several ones.[10]

Prior to multiplication, tape diagrams for addition and subtraction may be introduced in ways that show concrete numbers as lengths and suggest the connection between number as collection of objects and as a length. For example, *Mathematics for Elementary School*, a Japanese textbook series, shows a tape diagram for a second-grade addition problem in which each of two addends is shown as a line of dots within the tape. Similarly, for subtraction problem the subtrahend is shown as a line of dots within the tape (Hironaka et al., 2000/2006, pp. 13, 20).

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[10] Viewing whole-number multiplicands as many-as-one units changes the question "When to introduce a fraction times a collection of objects?" to "When to introduce a fraction times a whole number?" or "When to introduce a fraction times a many-as-one unit?" For discussion of the first question with regard to the Common Core State Standards, see http://mathematicalmusings.org/2011/08/12/drafty-draft-of-fractions-progression/#comment-1759.

The rule of like numbers for multiplication. When the multiplicand is a concrete number, the multiplicand and the multiplier are not like numbers. In that case, the product and the multiplicand are like numbers.

In a tape diagram that shows a multiplicand that is a concrete number, the multiplicand and product are depicted as lengths. (An example for third grade occurs in *Mathematics for Elementary School 3A,* p. 71.)

Division. If a product and one of the multiplicand or multiplier are known, the operation of finding the unknown multiplier or, respectively, multiplicand is called *division*. Division is also the operation of finding the unknown factor when the product and one factor are known.

To find an unknown multiplicand is called *partitive division*.

For a partitive division with a concrete-number dividend, a tape diagram shows the unknown multiplier as an unknown length. (An example for third grade occurs in *Mathematics for Elementary School 3B,* p. 13.)

To find an unknown multiplier is *quotitive division*.

For a quotitive division with a concrete-number dividend, a tape diagram shows the unknown multiplicand as an unknown number of lengths.

To find an unknown factor is neither quotitive nor partitive division.

Dividend, divisor, quotient, remainder. The known product in division is called the *dividend*. A known multiplicand, multiplier, or factor is called the *divisor*. The unknown, which is the result of the operation of division, is called the *quotient*.

The dividend may be the sum of a number smaller than the divisor and a product where one factor is the divisor. The former is called the *remainder.* In this case, the result of division has two parts: quotient and remainder. Remainder is a temporary term in school arithmetic. After fractions are introduced, there is no longer a need for this term.

The rule of like numbers for multiplication applied to division. In partitive division, dividend and quotient are like numbers. In quotitive division, dividend and divisor are like numbers.

VI.   Multiplication and division of fractions
(the definition of product is extended to fractions, axioms remain the same)

The product when the multiplier is a whole number. The *product* of two numbers is a third number which contains as many units as one number being taken as many times as the units in the other.

This definition is the same as that for whole numbers.

The product when the multiplier is a fraction. The *product* of a fraction and another number is a third number formed by partitioning the multiplicand into as many equal parts as the denominator of the multiplier and taking as many of those parts as the numerator of the multiplier.

Because a whole number can be considered as a fraction with denominator 1, this definition of product for a fractional multiplier reduces to the previous definition of product when this definition is used with a multiplier that is a whole number (identifying the whole number with a fraction with denominator 1). Thus, the same definition of product could be given whether the multiplier is a whole number or fraction.
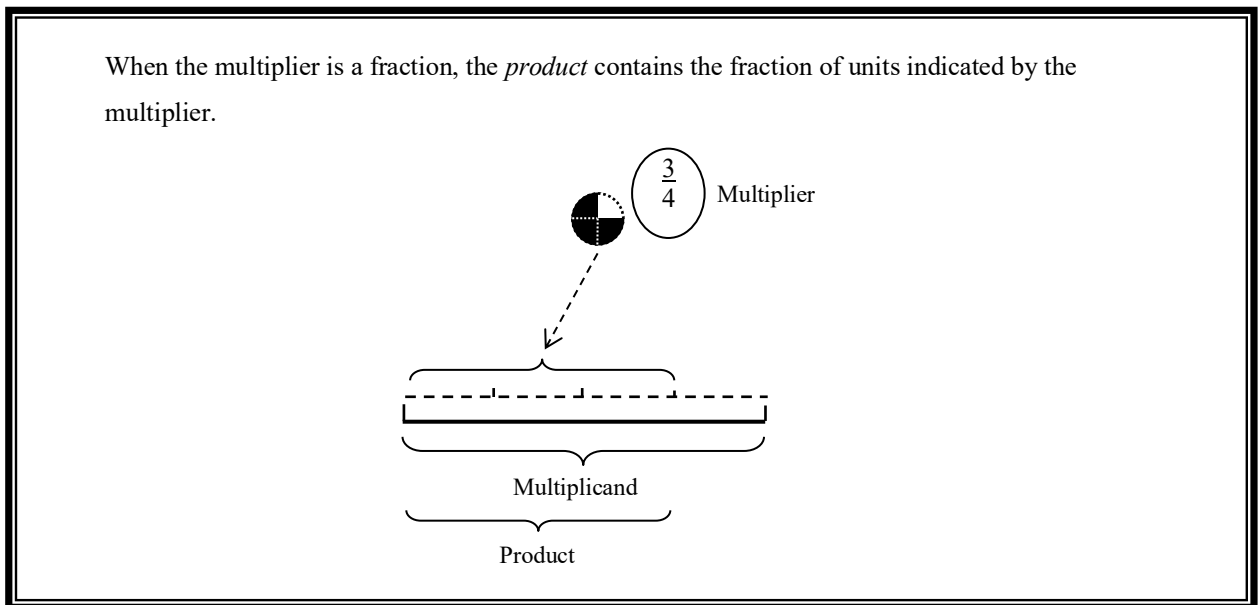

Fig. 7.1 The definition of the product when the multiplier is a fraction

The definition of multiplication when the multiplier is a fraction. To multiply a fractional unit times a whole number or fraction, consider the multiplicand as a many-as-one unit, partition it into equal parts according to the denominator of the multiplier. The product is equal to any one of those parts.

For example, to multiply ¼ times 5: Consider 5 as a many-as-one unit (as when 5 is the multiplicand in a product of two whole numbers). Partition it in 4 equal parts. Each part is equal to ¼ × 5.

For example, to multiply ¼ times 3/5: Consider 3/5 as a many-as-one unit. Partition it in 4 equal parts. Each part is equal to ¼ × 3/5.

To multiply several fractional units times a whole number or fraction. Consider the multiplicand as a many-as-one unit, partition it into equal parts according to the denominator of the multiplier. Take one of those parts the number of times indicated by the numerator.

For example, ¾ × 5 is 3 × (1/4 × 5), that is, ¼ × 5 taken 3 times.

A number multiplied by a fraction is to find the fractional part. For example, to find ¾ of 12, multiply 12 by ¾.

To find a fractional part of a number, use multiplication.

When the multiplier is a whole number different from 1, the result of multiplication is larger than the multiplicand. When the multiplier is a fraction smaller than 1, the result of multiplication is smaller than the multiplicand.

Whether the multiplicand is a whole number or a fraction does not change the definition of multiplication from its definition for whole numbers. For example, to find 12 times ¾ or "how many is twelve ¾?" in these problems the multiplicand is the fraction ¾ and the multiplier is the whole number 12. To be precise, the concept of multiplication expands only in the case when the multiplier is a fraction. For example, this occurs for: ¾ of 12 and "how many is ¾ of 8/5?" However, in teaching, we don't need to be as precise.

Historical notes. Here are examples of how multiplication by fractions has been described in textbooks.

What is the product of 7/8 by 5/6?

7/8 multiplied by *one* equals 7/8; hence 7/8 multiplied by 1/6 equals 1/6 of 7/8, which is 7/48, and 5/6 times 7/8 equals 5 times 7/48, which are 35/48. (Brooks, *Normal Higher Arithmetic*, 1877, p. 98)

In multiplying a given number by a fraction you take that fractional part of the given number. (Buckingham, 1953, p. 66)

Finding such a part of a number as is indicated by a fraction is called multiplying the number by the fraction. (Sheldon, 1886, p. 54)

Fraction and division. Division is the inverse of multiplication. As for whole numbers, it is to find an unknown multiplier or multiplicand. For example, we know that 6 ÷ 3 = 2 because

$2 \times 3 = 6$. Now that numbers include fractions and multiplication has been defined for fractions, quotients of whole numbers exist. For example, because $1/3 \times 3 = 1$, we know that the quotient of 1 and 3 is 1/3. Thus, the fraction 1/3 can be written as $1 \div 3$.

Similarly, we can write $2 \div 3$ as the fraction 2/3, $2 \div 5$ as 2/5, and $5 \div 9$ as 5/9.

A division expression is a fraction. One part of a partition of a unit into equal parts can be written as a division expression in which 1 is the dividend and the divisor is the number of parts. One part of a partition of a unit into equal parts can be written as a fraction. In the fraction, the dividend is called the *numerator*; the divisor is called the *denominator*. The division sign corresponds to the fraction bar.

Using only terms that students already know, we can also say whether or not the multiplier is a whole number or a fraction, the multiplicand (which can be a whole number or fraction) is considered as unit one. The multiplier decides whether the product is several of that unit or a fractional part of that unit.

a. The distance between Bill's house and school is 3 miles. The distance between Cathy's home and school is 3 times that distance. How far is Cathy's home from school?

b. The distance between Bill's house and school is 3 miles. The distance between Cathy's home and school is 2/3 of that distance. How far is Cathy's home from school?

c. The distance between Bill's house and school is 3/4 miles. The distance between Cathy's home and school is 3 times that distance. How far is Cathy's home from school?

d. The distance between Bill's house and school is 3/4 miles. The distance between Cathy's home and school is 2/3 of that distance. How far is Cathy's home from school?

In the four problems above, the distance between Bill's house and school is the multiplicand. The distance between Cathy's house and school is the product to be found. Whether or not the multiplicand is a whole number or a fraction, it is considered as a unit. Whether the product is several times the unit or a fraction of the unit is decided by whether the multiplier is a whole number or a fraction.

The definition of division with fractions derived from the definition of multiplication when the multiplier is a fraction. Division is the inverse operation of multiplication.

Quotitive division.

> To find what part of the second number is of the first, divide the second by the first. (Wentworth & Smith, 1915, p. 94)

As with whole number division, division is the inverse operation of multiplication. When the product and multiplicand or multiplier are known, to find the unknown multiplier or multiplicand.

Quotitive division is to find the multiplier. To find how many copies of a smaller number are contained in a given number or to find how many times larger than the smaller number the given number is. These examples illustrate conceptions of quotitive division in which the quotient is a whole number: How many twos are contained in 8? Eight is how many times larger than 2? Or many copies of 1/15 are contained in 1/3? One third is how many times larger than 1/15?

To find what fractional part of a larger number a smaller number is. These examples illustrate conceptions of quotitive division in which the quotient is a fraction: What fractional part of 8 is 2? Or, 2 is what fraction of 8? One fifteenth is what fraction of 1/3?

The two types of quotitive division are: A) multiplicand is smaller than or equal to multiplier; B) the quotient is a fraction less than 1.

The difference between these two types of quotitive division depends on which is larger: dividend or divisor. Whether or not the known numbers are whole numbers or fractions doesn't matter, however, this does not need to be made explicit to students.

Partitive division.

Given a part of a number, to find the number. (Wentworth & Smith, 1915, p. 95)

Partitive division is to find the multiplicand, when the product and the multiplier are known.

If the multiplier is a whole number this is to find the size of the many-as-one unit when the number of those units is known. (Partition 8 into four equal shares. How many in one share? Eight is four times a number, what is the number? Or, partition 9/17 into three equal shares. What size is a share?)

If the multiplier is a fraction, this to find the whole (the many-as-one unit) when a fractional part of the whole is known. (If a quarter of a number is 2, what is that number? Or, if 3/5 of a number is 1/3, what is that number?)

The difference between these two types of partitive division depends on which is larger: dividend or quotient. For the former, the quotient is smaller than the dividend. For the latter, the quotient is larger than the dividend.

The two types of partitive division: A) quotient is larger than or equal to dividend; B) divisor is a fraction less than 1.

Examples of problems about partitive division from a textbook:

If 2/3 of a number is 20, what is the number? (Sheldon, 1886, p. 60)

A farmer sold 5/8 of his farm for $4795, how much was the whole farm worth at that rate? (Sheldon, 1886, p. 73)

Partitive division of fractions contributed a new way of solving an old type of arithmetic problem. For example, there is a famous problem:

The head of a fish weighs 1/3 of the whole fish, his tail weighs 1/4 and his body weighs 30 ounces. What does the whole fish weigh?
(Sanford, 1927, p. 19)

Formerly, this was solved by the "Rule of False Position." If the weight of the whole fish is 12 ounces, then the head is 4 ounces, the tail is 3 ounces, and the body is 5 ounces. Evidently (as Sanford puts it), the weight of the whole fish is the same multiple of 12 ounces that 30 ounces is of 5 ounces. As Sanford notes, "forethought in selecting the guessed answer makes it possible to avoid fractions, at least until the last step of the work." Use of partitive division removes the need for this forethought.

To use partitive division to solve the problem, consider the weight of the whole fish as the unit. The known weight, 30 ounces, is obtained from the whole fish by cutting off its head and tail. The problem doesn't tell us what fraction of the whole the body is. However, we are told that the head is 1/3 and the tail is ¼. So the fractional part of the whole is: $(1-\frac{1}{3}-\frac{1}{4})$. Dividing the weight of the body by this fraction gives the weight of the fish in ounces:

$$30 \div (1-\frac{1}{3}-\frac{1}{4}) = 30 \div \frac{5}{12} = 72 .$$

Each term of the horizontal expression on the left-hand side of the equation corresponds to a quantity in the situation described in the problem. The solution method is described in a single expression.

Ma (2013) mentions the following problem:

Mrs. Chen made some tarts. She sold 3/5 of them in the morning and 1/4 of the remainder in the afternoon. If she sold 200 more tarts in the morning than in the afternoon, how many tarts did she make? (Curriculum Planning & Development Division, 1999, p. 70)

Like the fish problem, in this problem the whole (all the tarts) is unknown. We are told that a part of the whole is 200 tarts. The problem also tells us how to find what part of the whole the 200 tarts are. The 200 is the difference between the amount sold in the morning and the afternoon. The part sold in the morning is 3/5 of all the tarts. The part sold in the afternoon

is ¼ of the remainder which is $(1-\frac{3}{5}) \times \frac{1}{4}$. The 200 tarts is the difference between those sold in the morning minus those sold in the afternoon. Therefore, the total number of tarts is:

$$200 \div \lceil 3/5 - (1 - 3/5) \times 1/4 \rceil.$$

Compared with the fish problem, the tarts problem is more complicated because it involves multiplication of fractions.

The famous problem of Diophantus's tombstone is a problem of this type. With this approach, which is built on the definitions used in the theory of school arithmetic, students can even solve the problem of Diophantus's tombstone.

The rule of like numbers for multiplication and division of fractions is the same as that for whole numbers.

Summary

The theory has two sections. The first concerns whole numbers, and the second fractions. Important features of the first section are the definitions of number, unit, like number, sum, and product. In learning arithmetic with whole numbers, students' initial conceptions of each are established. In learning multiplication with whole numbers, students' conceptions of unit expand to include many-as-one unit.

The second section of the theory concerns fractions. Students' conceptions of unit expand again to include fractional unit. The underlying definition of number—a collection of like units—remains the same, but students' conceptions of number expand to include fractions and their conceptions of unit expand to include many-as-one fractional unit. The underlying definition of sum (and addition and subtraction) remains the same. In multiplication with a fractional multiplier, students use the conception of many-as-one fractional unit.

The notion of unit is used in measurement, in representing nonnegative integers and nonnegative rational numbers, in the definition of multiplication, and in solving word problems. This last use allows students the opportunity to formulate new units such as the weight of the fish in the fish problem and the total number of tarts in the tarts problem.

Concluding remarks

Instruction and instructional design. It is the role of instructional design and classroom instruction to help students learn the substance of the theory. This article has only briefly discussed how that might occur. (For more examples, see Ma, n. d., Ma & Kessel, 2018.) The role of the theory is to provide explicit definitions, explicit descriptions of relationships among concepts and explicit descriptions of how notation is used. In the pedagogical remarks, we have outlined how students' conceptions are intended to change with instruction.

Cognition. The parsimony of the theory may afford a cognitive payoff. The definitions, rules, and axioms of the theory illustrate how arithmetic knowledge may be organized via a small number of concepts and principles—a feature of expert knowledge (National Research Council, 2000, p. 38).

Coherence. Although the theory given in this article is in a form not meant for students, its definitions and rules are presented in an order that respects learning as well as logic. For example, the definition of unit is given before the definition of number because the latter depends on the former, but the definition of fractional unit is given after all definitions concerning whole numbers have been given although it could have been given earlier, because it depends only on the definition of unit. The order in which types of units are presented in the theory outlines how the concept of unit expands, increasing in abstraction from one thing as one unit, several units as one unit, a fraction of a unit as one unit, several fractional units as one unit. Introduction of the concept of many-as-one unit with whole numbers may help students to make the transition from the concept of number as cardinality to number as magnitude. The notion of unit can expand further, for instance, to include algebraic terms.

A second way in which the theory respects learning is in the asymmetry of multiplier and multiplicand in the definition of product. From a mathematical point of view, treating multiplier and multiplicand as interchangeable might seem simpler. However, the definition of product given in this article adheres more closely in structure to situations involving products that include concrete numbers. After students become familiar with multiplication, they can begin to view products more abstractly.

These are examples of how a theory can contribute to coherence in curricula and standards as described in the Common Core State Standards for Mathematics. The theory described in this article articulates:

> a sequence of topics . . . that are logical and reflect[s], where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. (Schmidt & Houang, 2002 as quoted in National Governors Association Center for Best Practices, Council of Chief State School Officers)

Acknowledgement

References

Anonymous. (1886). *Sheldons' complete arithmetic, with oral and written exercises*. New York & Chicago: Sheldon & Company.

Baker, A. (2016). Simplicity. In Edward N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), https://plato.stanford.edu/archives/win2016/entries/simplicity/

Bidwell, J., & Clason, R. (1970). *Readings in the history of mathematics education.* Washington, DC: National Council of Teachers of Mathematics. https://files.eric.ed.gov/fulltext/ED038316.pdf

Brooks, E. (1877). *Normal higher arithmetic.* Philadelphia: Sower, Potts & Co.

Buckingham, B. (1953). *Elementary arithmetic: Its meaning and practice*. Boston: Ginn and Company.

Curriculum Planning & Development Division (1999). *Primary Mathematics 5A Workbook* (3rd Ed.). Singapore: Ministry of Education.

Davies, C. (1834). *The common school arithmetic*. Hartford, Connecticut: H. F. Summer & Co.

Davies, C. (1857). *School arithmetic*. New York: A. S. Barnes & Co.

Hironaka et al. (2006/2000). *Mathematics for elementary school 2A*. Tokyo: Tokyo Shoseki. (M. Yoshida et al., trans.)

Ma, L. (2013). A critique of the structure of U.S. elementary school mathematics. *Notices of the American Mathematical Society*, *60*(10), 1282–1296, http://www.ams.org/notices/201310/fea-ma.pdf.

Ma, L. (n. d.). *One place addition and subtraction*, http://lipingma.net/math/One-place-number-addition-and-subtraction-Ma-Draft-2011.pdf.

Ma, L., & Kessel, C. (2018). The theory of school arithmetic: Whole numbers. In M. G. Bartolini Bussi & X. Sun (Eds.), *Building the foundation: Whole numbers in the primary grades*, ICMI study 23 (pp. 437–462). Cham, Switzerland: Springer International Publishing.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author.

National Research Council. (2000). *How people learn: Brain, mind, experience, and school: Expanded edition*. Washington, DC: The National Academies Press. https://doi.org/10.17226/9853.

Newell, D., & Tiesinga, E. (Eds.). (2019). *The international system of units (SI).* Gaithersburg, MD: National Institute of Standards and Technology.

Sanford, V. (1927). *The historical significance of certain standard problems in algebra* (Teachers College contributions to education, no. 251). New York: Teachers College.

Wentworth, G. & Smith, D. E. (1915). *Essentials of arithmetic*. Boston: Ginn and Company.

Young, J. W. A. (1920). *The teaching of mathematics in the elementary and secondary school* (2nd ed.)*.* New York: Longmans, Green and Co.